Natural Modes of Oscillation of Rotating Flexible Structures about Nontrivial Equilibrium

Leonard Meirovitch* and Jer-Nan Juang†
Virginia Polytechnic Institute and State University, Blacksburg, Va.

This paper is concerned with the vibrational characteristics of rotating flexible structures in the neighborhood of steady rotation. It is assumed that the structure undergoes deformations during the steady rotation, so that the equilibrium is nontrivial. The object of the paper is to formulate the eigenvalue problem associated with small oscillations of the structure about the nontrivial equilibrium and to develop efficient methods to compute the system natural frequencies and modal vectors. A numerical example showing the procedure for the calculation of spacecraft modes is presented.

Nomenclature

abc	= orbital axes
xyz	= spacecraft body axes
$\theta_j(t)$	= angular displacements
$v_i(x_i,t), w_i(x_i,t)$	= elastic displacements
$L(t)$	= Lagrangian functional
$\hat{L}_i(x_i,t)$	= Lagrangian density function
T_{0l}, T_{1l}, T_{2l}	= kinetic energy components
V_l	= potential energy
$\phi_j(x_i), \psi_j(x_i)$	= admissible functions
$\{q(t)\}$	= n-dimensional configuration vector
$[m]$	= $n \times n$ inertia matrix
$[g]$	= $n \times n$ gyroscopic matrix
$[k]$	= $n \times n$ stiffness matrix
$\{x(t)\}$	= $2n$-dimensional state vector
$[I]$	= $2n \times 2n$ matrix defined by Eqs. (23)
$[G]$	= $2n \times 2n$ matrix defined by Eqs. (23)
ω_r	= natural frequency of spacecraft
$\{y\}_r, \{z\}_r$	= natural modal vectors of spacecraft
$[K], [K']$	= $2n \times 2n$ matrices defined by Eqs. (27) and (30), respectively
A_0, B_0, C_0	= moments of inertia of rigid hub about x,y,z, respectively
ρ_i	= mass density of rod i
m_i	= tip mass of rod i
ℓ_i	= length of rod i
EI_{yi}, EI_{zi}	= flexural stiffnesses of rod i
h_{xi}, h_{yi}, h_{zi}	= distances from mass center to point of attachment of rod i
Ω	= spacecraft spin velocity and orbital velocity
α	= angle defining orientation of radial booms
β	= angle defining orientation of damper booms

Presented as Paper No. AAS-75-092 at the AAS/AIAA Astrodynamics Specialist Conference, Nassau, Bahamas, July 28-30, 1975; submitted December 12, 1974; revision received July 16, 1975. This paper presents one phase of research under NASA Research Grant NGR 47-004-098, sponsored by the Stabilization and Control Branch, Goddard Space Flight Center.

Index categories: Spacecraft Attitude Dynamics and Control; Structural Dynamic Analysis.

*Professor, Department of Engineering Science and Mechanics. Associate Fellow AIAA.

†Postdoctoral Research Associate, Department of Engineering Science and Mechanics. Currently Member of the Technical Staff, Computer Sciences Corporation, System Sciences Division, Silver Spring, Md.

I. Introduction

A problem of current interest is the dynamic behavior of flexible spacecraft rotating in space. In particular, the interest lies in the natural modes of oscillation of the complete vehicle, where natural modes are to be interpreted as including both the system natural frequencies and mode shapes. The concept of a modal vector that includes both the angular perturbations relative to a uniformly rotating frame and the elastic displacements relative to a given set of body axes is relatively new. References 1 and 2 introduced the concept and applied it to the development of a modal analysis for rotating structures.

In many cases, the oscillation takes place about an equilibrium in which the structure is deformed and the angular coordinates are different from zero; we shall refer to such equilibrium as nontrivial. Although this problem can be very important, it has received very little attention. Yet omission of this effect can lead to grossly erroneous results, particularly when the nontrivial equilibrium consists of large displacements. To illustrate the idea, let us consider an L-shaped bar fixed at one end and free at the other end and spinning uniformly with the angular velocity Ω about an axis through the fixed end. For simplicity, let us assume that the bar segment with the fixed end is normal to the spin axis and the segment with the free end is parallel to the spin axis. The centrifugal forces will cause the bar to undergo deformations proportional to Ω^2. The deformation pattern depends on space but not on time and it represents the state of nontrivial equilibrium. For large Ω or small bending stiffness, the equilibrium problem may require the solution of nonlinear differential equations. If the system is perturbed slightly and the equilibrium is stable, then the system will undergo small oscillations about the nontrivial deformed equilibrium and not about the trivial undeformed configuration. Note that the undeformed configuration is not an equilibrium state for this rotating system. This leads to a very important conclusion, namely, that the system can undergo large total displacements but the oscillations can remain small. It is easy to recognize that serious errors can occur if the total displacements are mistaken for oscillations, particularly for large nontrivial equilibrium.

Problems involving nontrivial equilibrium can arise in the case of gravity-gradient of spin-stabilized spacecraft with very flexible appendages that are not aligned with the spacecraft principal axes. Addressing himself to this problem, Flatley[3] obtained the nonlinear nontrivial equilibrium configuration of the Radio Astronomy Explorer (RAE) statellite by means of an analog computer. In deriving stability criteria for the RAE/B satellite, Meirovitch[4] obtained the nonlinear deformed equilibrium as a by-product, thus confirming the results of Ref. 3. Also inspired by the RAE satellite, Newton and Farrell[5] presented a method for the evaluation of the natural frequencies of a flexible gravity-gradient stabilized satellite oscillating about the deformed equilibrium. As generalized coordinates. Ref. 5 considered complete deformation patterns and angular displacements of the satellite. This approach tends to limit the number of degrees of freedom of the simulation. Moreover, one must guess in advance configuration patterns, which may not be possible for complicated structures. Hence, a more general approach is desirable.

This paper is concerned with the formulation of the eigenvalue problem for oscillation about nontrivial equilibrium, and with the development of efficient methods for the solution of the eigenvalue problem for high-order systems. The problem can be divided into four parts: a) derivaton of the nontrivial equilibrium; b) derivation of the variational equations of motion; c) system discretization and derivation of the eigenvalue problem; d) solution of the eigenvalue problem. The paper concentrates on the last three aspects of the problem. The derivation of the nontrivial equilibrium was presented in Ref. 4. It should be pointed out that, for modal analysis, one must have at his disposal not only the eigenvalues but also the eigenvectors of the system.

As an llusttion,the eigenvalue problem for a 30×30 matrix is solved. The mathematical model represents a 15 degree-of-freedom simulation of the RAE/B satellite.

II. Problem Formulation

We shall be concerned with the dynamic characteristics of a spacecraft consisting of a main rigid body with n flexible appendages. The appendages can be regarded as one-dimensional continua and, in addition to their own distributed mass, they possess concentrated masses at their tips. The mass center of the spacecraft is assumed to move in a circular orbit about a center of force fixed in an inertial space. Then denoting by abc a system of orital axes and identifying a system xyz with the body in undeformed state, the motion of axes xyz relative to axes abc can be described by three rotations $\theta_j(t)$ ($j=1,\ 2,\ 3$). Moreover, denoting by x_i ($i=1,2,\ldots,n$) the spatial variables associated with a typical elastic domain and by $v_i(x_i,t)$ and $w_i(x_i,t)$ the flexural displacements along the orthogonal axes y_i and z_i, respectively, we can write the Lagrangian in the general functional form[4]

$$L(t) = L_0(t) + \sum_{i=1}^{n} \left[\int_0^{\ell_i} \hat{L}_i(x_i,t)\,\mathrm{d}x_i + L_i(\ell_i,t) \right] \tag{1}$$

where

$$L_0(t) = L_0[\theta_j(t),\dot{\theta}_j(t)] \quad j=1,2,3 \tag{2a}$$

$$\hat{L}_i(x_i,t) = \hat{L}_i[\theta_j(t),\dot{\theta}_j(t),v_i(x_i,t),$$
$$\dot{v}_i(x_i,t),v_i'(x_i,t),v_i''(x_i,t),\dot{w}_i(x_i,t),\ldots,w_i''(x_i,t)]$$
$$i=1,2,\ldots,n \tag{2b}$$

$$L_i(\ell_i,t) = L_i[\theta_j(t),\dot{\theta}_j(t),v_i(\ell_i,t),\dot{v}_i(\ell_i,t),w_i(\ell_i,t),\dot{w}_i(\ell_i,t)]$$

in which L_0 is the Lagrangian corresponding to the system in undeformed state, $\hat{L}_i$ the Lagrangian density associated with any point of the elastic member i, and L_i the Lagrangian corresponding to the tip mass at $x_i = \ell_i$, where ℓ_i represents the length of the member i. Note that dots designate differentiations with respect to the time t and primes designate differentiations with respect to the spatial coordinate x_i.

III. Nontrivial Equilibrium

Equations (1) and (2) can be used in conjunction with Hamilton's principle to derive Lagrange's equations of motion about the trivial solution, namely, the solution for which all the rotational and elastic displacements and velocities are zero. These equations, however, are not very suitable when the system admits a nontrivial solution as an equilibrium state, where a nontrivial equilibrium is defined as a set of variables θ_j, v_i, w_i constant in time and satisfying Lagrange's equations. Indeed, it is shown in Ref. 6 that the nontrivial equilibrium is a solution of the equations

$$\frac{\partial L}{\partial \theta_j} = 0 \qquad j=1,2,3 \tag{3}$$

and

$$\frac{\partial \hat{L}_i}{\partial v_i} - \frac{\partial}{\partial x_i}\left[\frac{\partial \hat{L}_i}{\partial v_i'}\right] + \frac{\partial^2}{\partial x_i^2}\left[\frac{\partial \hat{L}_i}{\partial v_i''}\right] = 0, \quad 0<x_i<\ell_i$$
$$i=1,2,\ldots,n \tag{4a}$$

$$\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left[\frac{\partial \hat{L}_i}{\partial v_i''}\right] + \frac{\partial L_i}{\partial v_i}\right]\delta v_i = 0,\ (\partial \hat{L}_i/\partial v_i'')\delta v_i' = 0 \quad \text{at } x_i = \ell_i$$
$$i=1,2,\ldots,n \tag{4b}$$

$$-\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left[\frac{\partial \hat{L}_i}{\partial v_i''}\right]\right]\delta v_i = 0, \quad -(\partial L_i/\partial v_i'')\delta v_i' = 0 \quad \text{at } x_i = 0$$

as well as a set of equations similar to Eq. (4) for w_i. We shall denote the solutions of Eqs. (3) and (4), together with the set of equations for w_i, by θ_{j0}, $v_{i0}(x_i)$, $w_{i0}(x_i)$, where the first are constant and the latter are functions of the spatial variables alone.

IV. Perturbations about Equilibrium: Variational Equations of Motion

The interest lies in the oscillatory characteristics of the system about the nontrivial solution θ_{j0}, $v_{i0}(x_i)$, $w_{i0}(x_i)$. In particular, we wish to calculate the natural frequencies and the natural modes of oscillation of the entire spacecraft about the nontrivial equilibrium. To this end, we must first obtain Lagrange's equations of motion for small oscillations about that equilibrium. Denoting these oscillations by $\theta_{j1}(t)$, $v_{i1}(x_i,t)$, $w_{i1}(x_i,t)$ we can write

$$\theta_j(t) = \theta_{j0} + \theta_{j1}(t) \quad j=1,2,3 \tag{5a}$$

$$v_i(x_i,t) = v_{i0}(x_i) + v_{i1}(x_i,t)$$
$$w_i(x_i,t) = w_{i0}(x_i) + w_{i1}(x_i,t) \quad i=1,2,\ldots,n \tag{5b}$$

Inserting Eqs. (5) into Eq. (1), expanding a Taylor's series about θ_{j0}, $v_{i0}(x_i)$, $w_{i0}(v_i)$, and discarding constant terms as well as terms of order higher than two in the perturbations θ_{j1}, v_{i1}, w_{i1} and their time derivatives, we can write the perturbed Lagrangian in the form

$$L = T_{21} + T_{11} + T_{01} - V_1 \tag{6}$$

where

$$T_{21} = \tfrac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial^2 L}{\partial\dot{\theta}_{j0}\partial\dot{\theta}_{k0}}\dot{\theta}_{j1}\dot{\theta}_{k1}$$

$$
\begin{aligned}
&+\tfrac{1}{2}\sum_{i=1}^{n}\left\{\int_0^{\ell_i}\left[\frac{\partial^2\hat{L}_i}{\partial\dot{v}_{i0}^2}\dot{v}_{i1}^2+\frac{\partial^2 L_i}{\partial\dot{w}_{i0}^2}\dot{w}_{i1}^2\right.\right.\\
&+2\frac{\partial^2\hat{L}_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}}\dot{v}_{i1}\dot{w}_{i1}+2\sum_{j=1}^{3}\left[\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}}\dot{\theta}_{j1}\dot{v}_{i1}\right.\\
&\left.\left.+\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}}\dot{\theta}_{j1}\dot{w}_{i1}\right]\right]\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}^2}\dot{v}_{i1}^2+\frac{\partial^2 L_i}{\partial\dot{w}_{i0}^2}\dot{w}_{i1}^2\right.\\
&+2\frac{\partial^2 L_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}}\dot{v}_{i1}\dot{w}_{i1}+2\sum_{j=1}^{3}\left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}}\dot{\theta}_{j1}\dot{v}_{i1}\right.\\
&\left.\left.\left.+\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}}\dot{\theta}_{j1}\dot{w}_{i1}\right]\right]_{x_i=\ell_i}\right\}
\end{aligned}
\tag{7}
$$

is quadratic in the generalized velocities

$$
\begin{aligned}
T_{11}=&\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2 L}{\partial\theta_{j0}\partial\dot{\theta}_{k0}}\theta_{j1}\dot{\theta}_{k1}\\
&+\sum_{i=1}^{n}\left\{\int_0^{\ell_i}\left[\sum_{j=1}^{3}\left[\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial\dot{v}_{i0}}\theta_{j1}\dot{v}_{i1}\right.\right.\right.\\
&+\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{j0}\partial v_{i0}}\dot{\theta}_{j1}v_{i1}+\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial\dot{w}_{i0}}\theta_{j1}\dot{w}_{i1}\\
&\left.+\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{j0}\partial w_{i0}}\dot{\theta}_{j1}w_{i1}\right]+\frac{\partial^2\hat{L}_i}{\partial v_{i0}\partial\dot{w}_{i0}}v_{i1}\dot{w}_{i1}\\
&\left.+\frac{\partial^2\hat{L}_i}{\partial\dot{v}_{i0}\partial w_{i0}}\dot{v}_{i1}w_{i1}\right]\mathrm{d}x_i+\left[\sum_{j=1}^{3}\left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial\dot{v}_{i0}}\theta_{j1}\dot{v}_{i1}\right.\right.\\
&+\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial v_{i0}}\dot{\theta}_{j1}v_{i1}+\frac{\partial^2 L_i}{\partial\theta_{j0}\partial\dot{w}_{i0}}\theta_{j1}\dot{w}_{i1}\\
&\left.+\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial w_{i0}}\dot{\theta}_{j1}w_{i1}\right]+\frac{\partial^2 L_i}{\partial v_{i0}\partial\dot{w}_{i0}}v_{i1}\dot{w}_{i1}\\
&\left.\left.+\frac{\partial^2 L_i}{\partial\dot{v}_{i0}\partial w_{i0}}\dot{v}_{i1}w_{i1}\right]_{x_i=\ell_i}\right\}
\end{aligned}
\tag{8}
$$

is linear in the generalized velocities, and

$$
\begin{aligned}
T_{01}-V_1=&\tfrac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial^2 L}{\partial\theta_{j0}\partial\theta_{k0}}\theta_{j1}\theta_{k1}\\
&+\tfrac{1}{2}\sum_{i=1}^{n}\left\{\int_0^{\ell_i}\left[\frac{\partial^2\hat{L}_i}{\partial v_{i0}^2}v_{i1}^2+\frac{\partial^2\hat{L}_i}{\partial v_{i0}'^2}v_{i1}'^2\right.\right.\\
&+\frac{\partial^2\hat{L}_i}{\partial v_{i0}''^2}v_{i1}''^2+\frac{\partial^2\hat{L}_i}{\partial w_{i0}^2}w_{i1}^2+\frac{\partial^2\hat{L}_i}{\partial w_{i0}'^2}w_{i1}'^2\\
&+\frac{\partial^2 L_i}{\partial w_{i0}''^2}w_{i1}''^2\\
&+2\sum_{j=1}^{3}\left[\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial v_{i0}}\theta_{j1}v_{i1}+\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial w_{i0}}\theta_{j1}w_{i1}\right]\\
&+2\left[\frac{\partial^2\hat{L}_i}{\partial v_{i0}\partial w_{i0}}v_{i1}w_{i1}+\frac{\partial^2\hat{L}_i}{\partial v_{i0}'\partial v_{i0}''}v_{i1}'v_{i1}''\right.\\
&\left.\left.+\frac{\partial^2\hat{L}_i}{\partial w_{i0}'\partial w_{i0}''}w_{i1}'w_{i1}''\right]\right]\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial v_{i0}^2}v_{i1}^2+\frac{\partial^2 L_i}{\partial w_{i0}^2}w_{i1}^2\right.\\
&\left.\left.+2\sum_{j=1}^{3}\left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial v_{i0}}\theta_{j1}v_{i1}+\frac{\partial^2 L_i}{\partial\theta_{j0}\partial w_{i0}}\theta_{j1}w_{i1}\right]\right]_{x_i=\ell_i}\right\}
\end{aligned}
\tag{9}
$$

is free of generalized velocities.

In view of the above, the perturbed Lagrangian can be written in the general functional form

$$L=L(\theta_{j1},\dot{\theta}_{j1},v_{i1},v_{i1}',\ldots,w_{i1}',w_{i1}''),\ j=1,2,3\ ;\quad i=1,2,\ldots,n \tag{10}$$

Consequently, using Hamilton's principle, the variational equations can be written in the form of Lagrange's equations for the rotational motion

$$(\partial L/\partial\theta_{j1})-(\mathrm{d}/\mathrm{d}t)(\partial L/\partial\dot{\theta}_{j1})=0\quad j=1,2,3 \tag{11}$$

as well as the equations for the elastic motion

$$
\begin{aligned}
&\frac{\partial\hat{L}_i}{\partial v_{i1}}-\frac{\partial}{\partial t}\left[\frac{\partial\hat{L}_i}{\partial\dot{v}_{i1}}\right]-\frac{\partial}{\partial x_i}\left[\frac{\partial\hat{L}_i}{\partial v_{i1}'}\right]\\
&+\frac{\partial^2}{\partial x_i^2}\left[\frac{\partial\hat{L}_i}{\partial v_{i1}''}\right]=0,\quad 0<x_i<\ell_i,\quad i=1,2,\ldots,n
\end{aligned}
\tag{12a}
$$

where v_{i1} is subject to the boundary conditions

$$
\begin{aligned}
&\left[\frac{\partial\hat{L}_i}{\partial v_{i1}'}-\frac{\partial}{\partial x_i}\left[\frac{\partial\hat{L}_i}{\partial v_{i1}''}\right]\right.\\
&\left.+\frac{\partial L_i}{\partial v_{i1}}-\frac{\partial}{\partial t}\left[\frac{\partial L_i}{\dot{v}_{i1}}\right]\right]\delta v_{i1}=0\\
&\qquad\qquad\text{at } x_i=\ell_i\\
&\frac{\partial L_i}{\partial v_{i1}''}\delta v_{i1}'=0\qquad i=1,2,\ldots,n\\
&-\left[\frac{\partial\hat{L}_i}{\partial v_{i1}''}-\frac{\partial}{\partial x_i}\left[\frac{\partial\hat{L}_i}{\partial v_{i1}''}\right]\right]\delta v_{i1}=0\\
&\qquad\qquad\text{at } x_i=0\\
&-\frac{\partial\hat{L}_i}{\partial v_{i1}''}\delta v_{i1}'=0
\end{aligned}
\tag{12b}
$$

Equations similar in structure to Eqs. (12) can be written for w_{i1} by simply replacing v_{i1} by w_{i1}. It is perhaps worth reiterating that the above variational equations posses trivial equilibrium. Comparing Eqs. (11) and (12) with the ordinary Lagrange equations, Eqs. (35) and (36) of Ref. 6, we conclude that the variational equations have the same form as the ordinary Lagrange equations, but with the subscripts j and i replaced by $j1$ and $i1$. Hence the problem has been reduced to that of small oscillations about the trivial equilibrium of a structure having a new configuration, namely, the deformed configuration.

V. Discretization by Assumed-Modes Method

The variational equations discussed in the preceding section constitute a set of hybrid differential equations, in the sense that the equations for the rotational motion are ordinary differential equations and those for the elastic displacements are partial differential equations, where the latter are subject to given boundary conditions. It will prove convenient to transform the system into one consisting of ordinary differential equations alone. This can be done by using a discretization procedure based on the assumed-modes method (see Ref. 7, Sec. 6-5). Indeed, let us introduce the notation

$$\theta_{j1}(t)=q_j(t),\quad j=1,2,3 \tag{13a}$$

$$v_{11}(x_1,t)=\sum_{j=4}^{p+3}\phi_j(x_1)q_j(t)$$

$$w_{1l}(x_1,t)=\sum_{j=p+4}^{2p+3}\psi_j(x_1)q_j(t)$$

$$v_{2l}(x_2,t)=\sum_{j=2p+4}^{3p+3}\phi_j(x_2)q_j(t)$$

$$w_{2l}(x_2,t)=\sum_{j=3p+4}^{4p+3}\psi_j(x_2)q_j(t) \tag{13b}$$

. . . .

$$v_{nl}(x_n,t)=\sum_{j=2(n-1)p+4}^{(2n-1)p+3}\phi_j(x_n)q_j(t)$$

$$w_{nl}(x_n,t)=\sum_{j=(2n-1)p+4}^{2np+3}\psi_j(x_n)q_j(t)$$

where $\phi_j(x_i)$ and $\psi_j(x_i)$ are admissible functions, taken as the eigenfunctions of the fixed-base member. With this notation, Eq. (7) can be written in the matrix form

$$T_{2l}=\tfrac{1}{2}\{\dot q(t)\}^T[m]\{\dot q(t)\} \tag{14}$$

where $[m]$ is a constant symmetric matrix. On the other hand, Eq. (8) leads to the matrix form

$$T_{1l}=\{q(t)\}^T[f]\{\dot q(t)\} \tag{15}$$

where $[f]$ is a constant square matrix. Moreover, from Eq. (9), we can write

$$T_{0l}-V_l=-\tfrac{1}{2}\{q(t)\}^T[k]\{q(t)\} \tag{16}$$

where $[k]$ is a constant symmetric matrix. The elements of $[m]$, $[f]$, and $[k]$ are given in the Appendix.

Introducing Eqs. (14-16) into Eq. (6), we can write the Lagrangian in the matrix form

$$L=\tfrac{1}{2}\{\dot q\}^T[m]\{\dot q\}+\{q\}^T[f]\{\dot q\}-\tfrac{1}{2}\{q\}^T[k]\{q\} \tag{17}$$

Using the approach of Ref. 7 (see Sec. 3-4), we can write Lagrange's equations in the matrix form

$$(\mathrm{d}/\mathrm{d}t)\{\partial L/\partial\dot q\}-\{\partial L/\partial q\}=\{0\} \tag{18}$$

Hence inserting Eq. (17) into Eq. (18), we obtain the equations of motion

$$[m]\{\ddot q\}+([f]^T-[f])\{\dot q\}+[k]\{q\}=\{0\} \tag{19}$$

so that, introducing the notation

$$[g]=[f]^T-[f] \tag{20}$$

where $[g]$ is a skew symmetric matrix, $[g]^T=-[g]$, we obtain

$$[m]\{\ddot q\}+[g]\{\dot q\}+[k][q]=\{0\} \tag{21}$$

where $[m]$ is identified as the inertia matrix, $[g]$ is a "gyroscopic" matrix, and $[k]$ is a stiffness matrix which includes terms due to elastic, gravitational, and centrifugal effects.

VI. Natural Frequencies of the Complete Structure

The Liapunov direct method provides qualitative information concerning the stability or lack of stability of an equilibrium configuration. On the other hand, the variational equations (21) yield results of a more quantitative nature. Indeed, the associated eigenvalue problem yields the system natural frequencies and the normal modes for the complete structure, where the latter are defined later. It turns out that Eqs. (21) lead to an eigenvalue problem of a special nature. The nature of the eigenvalue problem can be conveniently discussed by converting the set of equations from second order to first order. Indeed, if the configuration vector $\{q(t)\}$ is of dimension N, then we can introduce the $2N$-dimensional state vector $\{x(t)\}$ in the form

$$\{x(t)\}=\begin{Bmatrix}\{\dot q(t)\}\\ \hdashline \{q(t)\}\end{Bmatrix} \tag{22}$$

No confusion should arise from denoting the state vector by $\{x(t)\}$, because the symbol x_i used to denote the position of a point in the elastic members represents a spatial coordinate independent of time and not a time-dependent generalized coordinate. Accordingly, if we introduce the $2N\times 2N$ matrices

$$[I]=\begin{bmatrix}[m] & [0]\\ [0] & [k]\end{bmatrix}\qquad [G]=\begin{bmatrix}[g] & [k]\\ -[k] & [0]\end{bmatrix} \tag{23}$$

then the set of N equations (21) can be transformed into a set of $2N$ first-order equations having the matrix form

$$[I]\{\dot x(t)\}+[G]\{x(t)\}=\{0\} \tag{24}$$

where $[I]$ is symmetric and $[G]$ is skew symmetric, $[I]=[I]^T$, $[G]=-[G]^T$, because $[m]$ and $[k]$ are symmetric and $[g]$ is skew symmetric.

The matrix equation (24) is of the special form treated in Ref. 1, so that the eigenvalue problem can be solved by the method developed there. Hence, letting $\{x(t)\}=e^{\lambda t}\{x\}$, where λ and $\{x\}$ are constant, we obtain the eigenvalue problem

$$\lambda[I]\{x\}+[G]\{x\}=\{0\} \tag{25}$$

It is shown in Ref. 1 that the solution of the eigenvalue problem (25) consists of $2N$ eigenvalues λ_r and eigenvectors $\{x\}_r$ $(r=1,2,\ldots,2N)$, where the eigenvalues consist of pairs of pure imaginary complex conjugates, $\lambda_r=\pm i\omega_r$, and the eigenvectors also consist of pairs of associated complex conjugates $\{x\}_r$ and $\{\bar x\}_r$ $(r=1,2,\ldots,N)$. Moreover, the eigenvectors are orthogonal in a certain sense. Reference 1 develops a method whereby the eigenvalue problem can be solved in terms of real quantities alone. The method is summarized in Sec. VII.

VII. A Method for the Solution of the Eigenvalue Problem

In solving the eigenvalue problem, Eq. (25), it is desirable to work with real quantities instead of complex quantities. Reference 1 develops a method of solution addressing itself to this very problem. We shall not repeat here the derivation details, but summarize instead the main features.

Assuming the solution $\lambda_r=i\omega_r$, $\{x\}_r=\{y\}_r+i\{z\}_r$, where $\{y\}_r$ and $\{z\}_r$ are the real and imaginary parts of the eigenvector, respectively, and separating the real and imaginary parts of the resulting equation, we obtain two algebraic equations that can be reduced to

$$\omega_r^2[I]\{y\}_r=[K][y]_r\quad \omega_r^2[I]\{z\}_r=[K]\{z\}_r\quad r=1,2,\ldots,N \tag{26}$$

where

$$[K]=[G]^T[I]^{-1}[G]=[K]^T \tag{27}$$

is a real symmetric matrix. From Eqs. (26), we conclude that the eigenvalue problem defined by $[I]$ and $[K]$ yields both the real part $\{y\}_r$ and the imaginary part $\{z\}_r$ of the eigenvector $\{x\}_r$ $(r=1,2,...,n)$. We shall refer to the problem (26) as being in *standard form.* Because the problem (26) is of order $2N$ it must possess $2N$ solutions. They consist of N pairs of repeated eigenvalues ω_r^2 and n pairs of associated eigenvectors $\{y\}_r$ and $\{z\}_r$ $(r=1,2,...,N)$. The eigenvectors $\{y\}_r$ and $\{z\}_r$ possess various properties that are not only interesting but also essential to various computational methods. The most important one is the orthogonality property, with the implied independence of the eigenvectors (Ref. 1).

Under certain circumstances, the eigenvalue problem, Eqs. (26), can be reduced to a more simple standard form, namely, a problem defined by a single symmetric matrix. Indeed, if $[I]$ is positive definite, we can introduce the linear transformations

$$\{y\}_r=[I]^{-1/2}\{y'\}_r \quad \{z\}_r=[I]^{-1/2}\{z'\}_r \quad r=1,2,...,n \tag{28}$$

into Eqs. (26), multiply the resulting equations on the left by $[I]^{-1/2}$, and obtain

$$\omega_r^2\{y'\}_r=[K']\{y'\}_r \quad \omega_r^2\{z'\}_r=[K']\{z'\}_r \quad r=1,2,...,n \tag{29}$$

where

$$[K']=[I]^{-1/2}[K][I]^{-1/2} \tag{30}$$

is a real symmetric matrix. Clearly the transformations (28) are possible only if $[I]^{-1/2}$ exists, which is ensured if $[I]$ is positive definite. Whereas Eqs. (29) yield the actual eigenvalues, the actual eigenvectors are obtained by inserting $\{y'\}_r$ and $\{z'\}_r$ back into transformations (28).

Introducing Eqs. (23) into Eq. (27), and recognizing that the multiplication of partitioned matrices and the inversion of block-diagonal matrices can be carried out as if the submatrices were single elements (provided the relative position of the elements is preserved), we can write

$$[K]=\begin{bmatrix}[g]^T & -[k]\\ [k] & [0]\end{bmatrix}\begin{bmatrix}[m]^{-1} & [0]\\ [0] & [k]^{-1}\end{bmatrix}\begin{bmatrix}[g] & [k]\\ -[k] & [0]\end{bmatrix}=\begin{bmatrix}[g]^T[m]^{-1}[g]+[k] & [g]^T[m]^{-1}[k]\\ [k][m]^{-1}[g] & [k][m]^{-1}[k]\end{bmatrix} \tag{31}$$

Moreover, using Eq.(30), we obtain

$$[K']=\begin{bmatrix}[m]^{-1/2} & [0]\\ [0] & [k]^{-1/2}\end{bmatrix}\begin{bmatrix}[g]^T[m]^{-1}[g]+[k] & [g]^T[m]^{-1}[k]\\ [k][m]^{-1}[g] & [k][m]^{-1}[k]\end{bmatrix}\times\begin{bmatrix}[m]^{-1/2} & [0]\\ [0] & [k]^{-1/2}\end{bmatrix}$$

$$=\begin{bmatrix}[A]^T[A]+[B]^T[B] & [A]^T[B]^T\\ [B][A] & [B][B]^T\end{bmatrix} \tag{32}$$

where

$$[A]=[m]^{-1/2}[g][m]^{-1/2}, \quad [B]=[k]^{1/2}[m]^{-1/2} \tag{33}$$

Note that, to produce the matrix $[K']$, it is only necessary to obtain $[m]^{-1/2}$ and $[k]^{1/2}$. This involves solving the eigenvalue problems associated with $[m]$ and $[k]$, respectively.

A large variety of algorithms for the solution of the eigenvalue problem associated with a real symmetric matrix can be found in many texts on numerical analysis, such as that by Ralston[8] and that by Wilkinson.[9] Useful algorithms include the power method, Jacobi's method, Given's method, Householder's method, the QR algorithm, and inverse iteration. The experience of these authors with Jacobi's method indicates that, in practice, the existence of very close and multiple eigenvalues affects the number of iterations required to reduce the off-diagonal elements below a prescribed level to a lesser extent than one might expect. For example, in the case of matrices of order up to 50 and for word lengths of 32 to 48 binary digits, the total number of sweeps has averaged about 5 to 6, and the number of iterations required on matrices having multiple eigenvalues has often proved to be somewhat lower than the average. Moreover, even when the matrix has some pathologically close eigenvalues, the corresponding eigenvectors are almost exactly orthogonal. It is evident that they span the correct subspace and give full digital information. This is, perhaps, the most satisfactory feature of Jacobi's method. Moreover, the extreme simplicity of this method from the programing point of view makes it very desirable. The results presented in Sec. VIII were obtained by this method. It should be stressed, however, that the solution of the eigenvalue problem for $[K']$ is not restricted in any way to Jacobi's method and, indeed, it can be obtained by any other method, such as one of those listed above, or a combination thereof.

VIII. Illustrative Example: The RAE/B Satellite

The general theory developed has been used to calculate the natural frequencies and natural modes of oscillation about nontrivial equilibrium of the RAE/B satellite. The satellite, shown in Fig. 1, consists of a rigid hub with two pairs of cantilevered radial booms (numbered 1, 2, 3, 4) possessing tip masses, and one pair of cantilevered damper booms‡ (numbered 5, 6). The spacecraft is gravity-gradient stabilized. The analysis consists of: a) the determination of the nontrivial equilibrium, b) the calculation of the matrix coefficient m_{ij}, g_{ij}, and k_{ij}, and c) the solution of the eigenvalue problem. Explicit expressions for the nontrivial equilibrium and the perturbed Lagrangian for the RAE/B satellite have been derived in Ref. 6, so they will be omitted here for brevity. The parameters used are as follows

$A_0=87.74$ slug ft,^{2} $\quad B_0=83.74$ slug ft,^{2} $\quad C_0=18$ slug ft,^{2}

$\rho_1=\rho_2=\rho_3=\rho_4=4.348\times10^{-4}$ slug ft^{-1}

$\rho_5=\rho_6=4.596\times10^{-4}$ slug ft^{-1}

$m_1=m_2=m_3=m_4=2.40\times10^{-3}$ slug, $\quad m_5=m_6=0$

$\ell_1=\ell_2=\ell_3=\ell_4=750$ ft, $\quad \ell_5=\ell_6=315$ ft

$EI_{y1}=EI_{z1}=EI_{y2}=\ldots=EI_{z4}=15.278$ lb ft,^{2} $\quad \alpha=30°$

$EI_{y5}=EI_{z5}=EI_{y6}=EI_{z6}=13.889$ lb ft,^{2} $\quad \beta=25°$

$h_{x1}=h_{x4}=0.973$ ft $\quad h_{x2}=h_{x3}=0.878$ ft $\quad h_{x5}=h_{x6}=0$

$h_{y1}=-h_{y4}=0.705$ ft, $\quad h_{y2}=-h_{y3}=-0.760$ ft,

‡Note that the damper booms in the actual RAE/B satellite are supported by torsional springs instead of being cantilevered. The difference in the system frequencies for the two slightly different mathematical models should not be significant. It should also be noted that Ref. 5 assumes that the damper booms are rigid. This permits a 12-degree-of-freedom simulation.

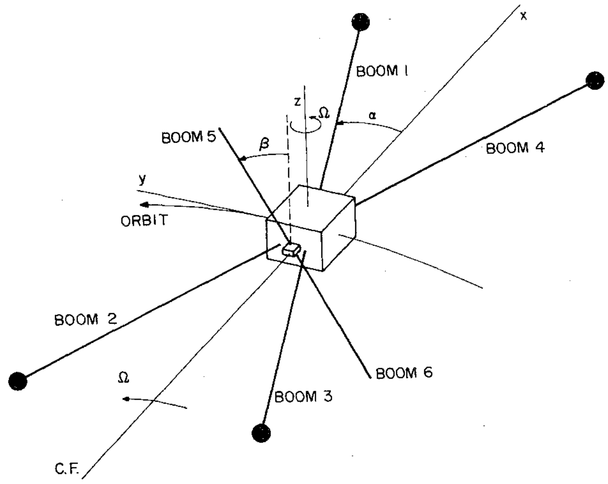

Fig. 1 Radio astronomy explorer-lunar (RAE/B) satellite.

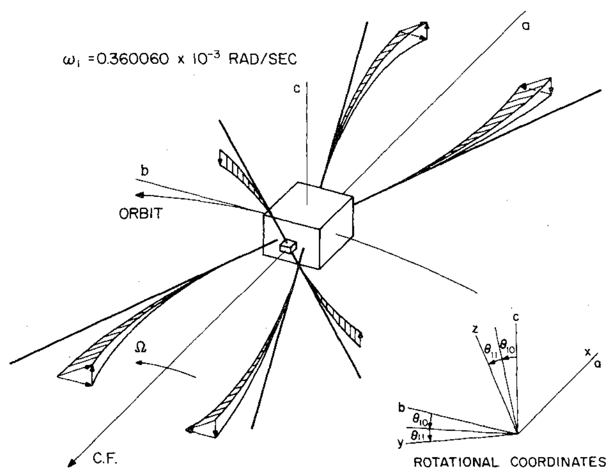

Fig. 2 Satellite first mode.

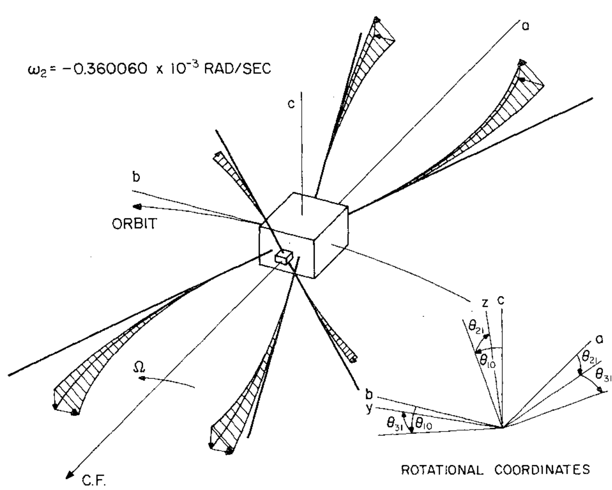

Fig. 3 Satellite second mode.

$$h_{y5} = h_{y6} = -1.800 \text{ ft}$$

$$h_{z1} = h_{z2} = h_{z3} = h_{z4} = h_{z5} = h_{z6} = 0$$

$$\Omega = 4.653 \times 10^{-4} \text{ rad sec}^{-1}$$

A. Nontrivial Equilibrium

To evaluate the elastic displacements $v_{i0}(x_i)$ and $w_{i0}(x_i)$ $(i=1,2,\ldots,6)$ we assume the solution of Eqs. (3) and (4) in the form

$$v_{i0}(x_i) = \sum_{r=1}^{p} a_{ri}\phi_r(x_i) \qquad w_{i0}(x_i) = \sum_{r=1}^{p} b_{ri}\phi_r(x_i) \qquad i=1,2,\ldots,n \tag{34}$$

where

$$\phi_r(x_i) = A_r[(\cos\beta_r\ell_i + \cosh\beta_r\ell_i)(\sin\beta_r x_i - \sinh\beta_r x_i) - (\sin\beta_r\ell_i + \sinh\beta_r\ell_i)(\cos\beta_r x_i - \cosh\beta_r x_i)] \tag{35}$$

are eigenfunctions corresponding to a bar in bending with the end $x_i = 0$ fixed and having a mass m_i attached at the end $x_i = \ell_i$. The eigenvalues $\beta_r\ell_i$ are solutions of the characteristic equation

$$(1 + \cos\beta_r\ell_i\cosh\beta_r\ell_i) = \beta_r\ell_i(m_i/\rho_i\ell_i)(\sin\beta_r\ell_i\cosh\beta_r\ell_i - \cos\beta_r\ell_i\sinh\beta_r\ell_i) \tag{36}$$

Moreover, the amplitudes A_r are such that the eigenfunctions $\phi_r(x_i)$ are orthonormal, i.e., they satisfy the relation

$$\int_0^{\ell_i} \rho_i\phi_r(x_i)\phi_s(x_i)\,dx_i + m_i\phi_r(\ell_i)\phi_s(\ell_i) = \delta_{rs} \tag{37}$$

where δ_{rs} is the Kronecker delta. Limiting the series in Eqs. (34) to two terms, $p=2$, the first two roots of Eq. (36) and the amplitudes A_r corresponding to $i=1, 2, 3, 4$ are

$\beta_1\ell_i = 1.86154 \qquad A_1 = 0.425880 \text{ slug}^{-1/2}$

$\beta_2\ell_i = 4.66095 \qquad A_2 = 0.033648 \text{ slug}^{-1/2}$

The first two roots of Eq. (36) with $m_i = 0$, and the amplitudes A_1 and A_2 corresponding to $i=5, 6$ are

$\beta_1\ell_i = 1.87511 \qquad A_1 = 0.63510 \text{ slug}^{-1/2}$

$\beta_2\ell_i = 4.69414 \qquad A_2 = 0.04899 \text{ slug}^{-1/2}$

By using the technique developed in Ref. 6, we obtain the final results given in Table 1.

It will prove of interest to list the elastic displacements of the end points. These displacements are

$v_{10}(\ell_1)$	$= -140.95$ ft	$w_{10}(\ell_1)$	$= 1.8990$ ft
$v_{20}(\ell_2)$	$= 140.92$ ft	$w_{20}(\ell_2)$	$= 1.8993$ ft
$v_{30}(\ell_3)$	$= -140.92$ ft	$w_{30}(\ell_3)$	$= -1.8992$ ft
$v_{40}(\ell_4)$	$= 140.95$ ft	$w_{40}(\ell_4)$	$= -1.8990$ ft
$v_{50}(\ell_5)$	$= -0.048677$ ft	$w_{50}(\ell_5)$	$= 0.86067$ ft
$v_{60}(\ell_6)$	$= -0.048669$ ft	$w_{60}[\ell_6)$	$= 0.86067$ ft

B. Calculation of the Matrix Coefficients

Using Eqs. (A1)-(A19) and Eq. (20), in conjunction with the above data, we obtain the elements of the matrices $[m]$, $[g]$, and $[k]$. Lack of space prevents listing the matrix elements. Then, using Eq. (32), we obtain the matrix $[K']$. The elastic displacements, measured from equilibrium, were represented by only one admissible function each, resulting in two degrees of freedom per boom. Experience has shown that the addition of another admissible function yields only a small change in the system natural frequencies.

C. Solution of the Eigenvalue Problem

Solving the eigenvalue problem associated with $[K']$ by Jacobi's method, we obtain the natural frequencies squared given in Table 2.

Table 1 Constants defining nontrivial equilibrium

$\theta_{10} = 1.40868 \times 10^{-1}$ rad $= 8.07115°$
$\theta_{20} = -3.42313 \times 10^{-8}$ rad $= -1.96131 \times 10^{-6}°$
$\theta_{30} = 1.34899 \times 10^{-6}$ rad $= 7.72913 \times 10^{-5}°$

i	a_{1i}	a_{2i}	b_{1i}	b_{2i}
1	-0.41093×10^2	-0.30339	0.55781	0.838534×10^{-2}
2	0.41086×10^2	0.30324	0.55789	0.838537×10^{-2}
3	-0.41085×10^2	-0.30324	-0.55788	$-0.8\ 8536 \times 10^{-2}$
4	0.41093×10^2	0.30339	-0.55781	-0.838534×10^{-2}
5	-0.93895×10^{-2}	-0.12895×10^{-3}	0.16439	0.65280×10^{-3}
6	-0.93881×10^{-2}	-0.12894×10^{-3}	0.16439	0.65280×10^{-3}

Table 2 Natural frequencies squared

i	$\omega_i^2 \times 10^6$	i	$\omega_i^2 \times 10^6$
1	1.29648×10^{-1}	2	1.29648×10^{-1}
3	4.57848×10^{-1}	4	4.57848×10^{-1}
5	8.02608×10^{-1}	3	8.02608×10^{-1}
7	1.44713	8	1.44713
9	1.44713	10	1.44713
11	1.46268	12	1.46268
13	1.94639	14	1.94639
15	2.00414	16	2.00414
17	2.14185	18	2.14185
19	2.14185	20	2.14185
21	1.34030×10	22	1.34030×10
23	2.4113×10	24	2.41134×10
25	3.68120×10	26	3.68120×10
27	3.77371×10	28	3.77371×10
29	1.25320×10^2	30	1.25230×10^2

The modal vectors have also been obtained, but lack of space does not permit their listing. As a matter of interest, however, the first two modes are plotted in Figs. 2 and 3. Because these two modes correspond to the same eigenvalue ω_i^2, any linear combination of the two modes is also a mode (see Ref. 1). This fact can be used at times to simplify the vectors.

IX. Conclusions

This paper presents a method for the determination of the natural frequencies and mode shapes of rotating structures oscillating about nontrivial equilibrium. The formulation of the eigenvalue problem involves the derivation of the variational equations about the nontrivial equilibrium and the discretization of these equations. The linearized discrete system is defined by two symmetric matrices and one skew symmetric matrix. Hence, it belongs to the class of linear gyroscopic systems. Using a technique developed by Meirovitch[1] the eigenvalue problem is transformed into one defined by a single real symmetric matrix, the solution of which can be obtained by a large variety of algorithms. The approach is suitable for the solution of high-order systems.

As an illustration, the problem associated with a 15-degree-of-freedom system is solved. Both natural frequencies and modal vectors for the complete structure are obtained. The mathematical model simulates the RAE/B satellite, a gravity-gradient stabilized flexible spacecraft.

Appendix

The elements of the matrices $[m]$, $[f]$, and $[k]$ are as follows

$$m_{jk} = \partial^2 L / \partial\dot{\theta}_{j0} \partial\dot{\theta}_{k0} \quad j,k=1,2,3 \tag{A1}$$

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}} \phi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}} \phi_k(x_i) \right]_{x_i=\ell_i}$$

$j=1,2,3;\ k=2(i-1)p+4,\ 2(i-1)p+5,\dots,(2i-1)p+3$
$i=1,2,\dots,n$ (A2)

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}} \psi_k(x_i)\,\mathrm{d}x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}} \psi_k(x_i) \right]_{x_i=\ell_i}$$

$j=1,2,3;\ k=(2i-1)p+4,\ (2i-1)p+5,\dots,2ip+3$
$i=1,2,\dots,n$ (A3)

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{v}_{i0}^2} \phi_j(x_i)\phi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}^2} \phi_j(x_i)\phi_k(x_i) \right]_{x_i=\ell_i}$$

$j,k=2(i-1)p+4, 2(i-1)p+5,\dots,(2i-1)p+3;\ i=1,2,\dots,n$ (A4)

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i) \right]_{x_i=\ell_i}$$

$j=2(i-1)p+4, 2(i-1)p+5,\dots,\ (2i-1)p+3$

$k=\ (2i-1)p+4,\ (2i-1),\ p+5,\dots,2ip+3$

$i=1,2,\dots,n$ (A5)

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{w}_{i0}^2} \psi_j(x_i)\psi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\dot{w}_{i0}^2} \psi_j(x_i)\psi_k(x_i) \right]_{x_i=\ell_i}$$

$j,k=\ (2i-1)p+4,\ (2i-1)p+5,\dots,2ip+3;\ i=1,2,\dots,n$ (A6)

$$f_{jk} = \frac{\partial^2 L}{\partial\theta_{j0}\partial\dot{\theta}_{k0}}, \quad j,k=1,2,3 \tag{A7}$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\theta_{j0}\partial\dot{v}_{i0}} \phi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial\dot{v}_{i0}} \phi_k(x_i) \right]_{x_i=\ell_i}$$

$j=1,2,3;\ k=2(i-1)p+4,\ 2(i-1)p+5,\dots,\ (2i-1)p+3$

$i=1,2,\dots,n$ (A8)

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\theta_{j0}\partial\dot{w}_{i0}} \psi_k(x_i)\,\mathrm{d}x_i + \left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial\dot{w}_{i0}} \psi_k(x_i) \right]_{x_i=\ell_i}$$

$j=1,2,3;\ k=\ (2i-1)p+4,\ (2i-1)p+5,\dots,\ 2ip+3$

$$i=1,2,\ldots,n \tag{A9}$$

$$f_{jk}=\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{k0}\partial v_{i0}}\phi_j(x_i)\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{k0}\partial v_{i0}}\phi_j(x_i)\right]_{x_i=\ell_i}$$

$$j=2(i-1)p+4,\ 2(i-1)p+5,\ldots,\ (2i-1)p+3$$

$$i=1,2,\ldots,n;\ k=1,2,3 \tag{A10}$$

$$f_{jk}=\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial\dot{\theta}_{k0}\partial w_{i0}}\psi_j(x_i)\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{k0}\partial w_{i0}}\psi_j(x_i)\right]_{x_i=\ell_i}$$

$$j=(2i-1)p+4,\ (2i-1)p+5,\ldots,\ 2ip+3$$

$$i=1,2,\ldots,n;\ k=1,2,3 \tag{A11}$$

$$f_{jk}=\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial v_{i0}\partial\dot{w}_{i0}}\phi_j(x_i)\psi_k(x_i)\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial v_{i0}\partial\dot{w}_{i0}}\phi_j(x_i)\psi_k(x_i)\right]_{x_i=\ell_i}$$

$$j=2(i-1)p+4,\ 2(i-1)p+5,\ldots,\ (2i-1)p+3$$

$$k=(2i-1)p+4,\ (2i-1)\ p+5,\ldots,2ip+3$$

$$i=1,2,\ldots,n \tag{A12}$$

$$f_{jk}=\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial\dot{v}_{i0}\partial w_{i0}}\phi_k(x_i)\psi_j(x_i)\mathrm{d}x_i+\left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}\partial w_{i0}}\phi_k(x_i)\psi_j(x_i)\right]_{x_i=\ell_i}$$

$$j=(2i-1)p+4,(2i-1)p+5,\ldots,\ 2ip+3,$$

$$k=2(i-1)p+4,\ 2(i-1)p+5,\ldots,\ (2i-1)p+3$$

$$i=1,2,\ldots,n \tag{A13}$$

$$k_{jk}=-\frac{\partial^2 L}{\partial\theta_{j0}\partial\theta_{k0}}\qquad j,k=1,2,3 \tag{A14}$$

$$k_{jk}=-\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial v_{i0}}\phi_k(x_i)\mathrm{d}x_i-\left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial v_{i0}}\phi_k(x_i)\right]_{x_i=\ell_i}$$

$$j=1,2,3;\ k=2(i-1)p+4,\ 2(i-1)p+5,\ldots,\ (2i-1)p+3$$

$$i=1,2,\ldots,n \tag{A15}$$

$$k_{jk}=-\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial\theta_{j0}\partial w_{i0}}\psi_k(x_i)\mathrm{d}x_i-\left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial w_{i0}}\psi_k(x_i)\right]_{x_i=\ell_i}$$

$$j=1,2,3;\ k=(2i-1)p+4,\ (2i-1)p+5,\ldots,\ 2ip+3$$

$$i=1,2,\ldots,n \tag{A16}$$

$$k_{jk}=-\int_0^{\ell_i}\left\{\frac{\partial^2\hat{L}_i}{\partial v_{i0}^2}\phi_j(x_i)\phi_k(x_i)\phi_k(x_i)+\frac{\partial^2\hat{L}_i}{\partial v_{i0}'^2}\phi_j'(x_i)\phi_k'(x_i)+\frac{\partial^2\hat{L}_i}{\partial v_{i0}''^2}\phi_j''(x_i)\phi_k''(x_i)+\frac{\partial^2\hat{L}_i}{\partial v_{i0}'\partial v_{i0}''}\times\left[\phi_j'(x_i)\phi_k''(x_i)+\phi_j''(x_i)\phi_k'(x_i)\right]\right\}\mathrm{d}x_i-\left[\frac{\partial^2 L_i}{\partial v_{i0}^2}\phi_j(x_i)\phi_k(x_i)\right]_{x_i=\ell_i}$$

$$j,k=2(i-1)p+4,\ 2(i-1)p+5,\ldots,(2i-1)p+3$$

$$i=1,2,\ldots,n \tag{A17}$$

$$k_{jk}=-\int_0^{\ell_i}\frac{\partial^2\hat{L}_i}{\partial v_{i0}\partial w_{i0}}\phi_j(x_i)\psi_k(x_i)\mathrm{d}x_i-\left[\frac{\partial^2 L_i}{\partial v_{i0}\partial w_{i0}}\phi_j(x_i)\psi_k(x_i)\right]_{x_i=\ell_i}$$

$$j=2(i-1)p+3\ k=(2i-1)p+4,\ (2i-1)p+5,\ldots,2ip+3$$

$$i=1,2,\ldots,n \tag{A18}$$

$$k_{jk}=-\int_0^{\ell_i}\left\{\frac{\partial^2\hat{L}_i}{\partial w_{i0}^2}\psi_j(x_i)\psi_k(x_i)+\frac{\partial^2\hat{L}_i}{\partial w_{i0}'^2}\psi_j'(x_i)\psi_k'(x_i)+\frac{\partial^2\hat{L}_i}{\partial w_{i0}''^2}\psi_j''(x_i)\psi_k''(x_i)+\frac{\partial^2\hat{L}_i}{\partial w_{i0}'\partial w_{i0}''}\times\left[\psi_j'(x_i)\psi_k''(x_i)+\psi_j''(x_i)\psi_k'(x_i)\right]\right\}\mathrm{d}x_i-\left[\frac{\partial^2 L_i}{\partial w_{i0}^2}\psi_j(x_i)\psi_k(x_i)\right]_{x_i=\ell_i}$$

$$j,k=(2i-1)p+4,\ (2i-1)p+5,\ldots,\ 2ip+3$$

$$i=1,2,\ldots,n \tag{A19}$$

References

[1]Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal,* Vol. 12, Oct. 1974, pp. 1337-1342.

[2]Meirovitch, L., "A New Modal Method for the Response of Structures Rotating in Space," Presented as Paper 74-002 at the 25th International Astronautical Congress of the I.A.F., Amsterdam, The Netherlands, Sept. 30-Oct. 5, 1974. *Acta Astronautica,* Vol. 2, July-Aug. 1975, pp. 563-576.

[3]Flatley, T.W., "Equilibrium Shapes of an Array of Long Elastic Structural Members in Circular Orbit," NASA TN D-3173, March 1966.

[4]Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems in the Neighborhood of Nontrivial Equilibrium," *AIAA Journal,* Vol. 12, July 1974, pp. 889-898.

[5]Newton, J.K. and Farrell, J.L., "Natural Frequencies of a Flexible Gravity-Gradient Satellite," *Journal of Spacecraft and Rockets,* Vol. 5, May 1968, pp. 560-569.

[6]Meirovitch, L. and Juang, J.N., "Dynamics of a Gravity-Gradient Stabilized Flexible Spacecraft," NASA CR-2456, Aug. 1974.

[7]Meirovitch, L., *Analytical Methods in Vibrations,* Macmillan, New York, 1967.

[8]Ralston, A., *A First Course in Numerical Analysis,* McGraw-Hill, New York, 1965.

[9]Wilkinson, J.H., *The Algebraic Eigenvalue Problem,* Oxford, New York, 1965.